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Relativistic Internal Time Operator

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We construct a self-adjoint time operator for massless relativistic systems in terms of the generators of the Poincaré group. The Lie algebra generated by the time operator and the generators of the Poincaré group turns out to be an infinite-dimensional extension of the Poincaré algebra. The internal time operator generates two new entities, namely the velocity operator and the internal position operator. The transformation properties of the internal time and position operator under Lorentz boosts are different from what one would expect from relativity theory. This difference reflects the fact that the time concept associated with the internal time operator is radically different from the time coordinate of Minkowski space, due to the nonlocality of the time operator. The spectral projections of the time operator allow us to construct incoming subspaces for the wave equation without invoking Huygens' principle, as in two and one spatial dimensions where Huygens' principle does not hold.

1. INTRODUCTION

Consider the unitary evolution group U_t, $t \in R$, on a separable Hilbert space $\mathscr{H}$. An internal time operator T for U_t is a self-adjoint operator T with domain $\mathscr{D}$ on which the following property holds:

$$U_{-t} T U_t = T + tI \tag{1}$$

Relation (1) means, of course, that the domain $\mathscr{D}$ is invariant under the unitary group U_t.

The internal time operator is canonically conjugate to the anti-self-adjoint generator P_0 of the unitary group $U_t = e^{P_0 t}$:

$$[P_0, T] = -I \tag{2}$$

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The time operator T allows us to attribute the average age $\langle y, Ty \rangle$ to the states $y \in \mathscr{H}$. The average age of the evolved state $U_t y$ advances in step with the external clock time t:

$$\langle U_t y, TU_t y \rangle = \langle y, Ty \rangle + t \tag{3}$$

Relation (3) is equivalent to the internal time property (1).

Internal time operators for unitary dynamics were introduced by Misra (1978) in the context of unstable Kolmogorov dynamical systems. The unitary group U_t defines the evolution of densities in the Liouville space and the spectral projections of the time operator are the conditional expectations over the time-evolved K-partition. These dynamical systems allow for an exact passage to irreversible Markov processes through nonunitary intertwining transformations (Goldstein *et al.*, 1981; Misra and Prigogine, 1982; Prigogine, 1982). Similarly, quantum systems admit time operators on the Liouville space of density operators if the Hamiltonian has absolutely continuous spectrum (Misra *et al.*, 1979). Time operators were also introduced for relativistic fields (Misra, 1987; Antoniou, 1988; Misra and Antoniou, 1988; Antoniou and Misra, 1989). The existence of time operators for relativistic fields implies that the relativistic fields are Kolmogorov dynamical systems.

In this paper we study the time operator for relativistic fields and its transformation laws under Lorentz boosts, which turn out to be radically different from what is expected by special relativity. Naturally, we are led to the Lie algebra generated by the time operator and the ten generators of the Poincaré group. This algebra, called the relativistic internal time algebra, turns out to be an infinite-dimensional extension of the Poincaré algebra. The internal time gives rise to the velocity observable and to an internal position observable. Since the transformation laws of the internal time and position operators under Lorentz boosts are different from the corresponding formulas of Einstein, the internal space-time associated with the time and position operators is not the Minkowski space-time of events labels of localized observations. The algebra of relativistic systems with internal time generates nonlocal symmetries and shows that the simple wave equation is not only invariant under the 15-parameter conformal group, but also under an infinite-parameter group of nonlocal transformations.

2. A TIME OPERATOR FOR THE WAVE EQUATION

Consider the solutions of the wave equation:

$$\partial_t^2 \psi = \Delta \psi \tag{4}$$

in the Fourier representation

$$\psi(x)=\int d(k)\, e^{-ikx}\tilde{\psi}(x) \qquad k_0^2\tilde{\psi}=|\mathbf{k}|^2\tilde{\psi} \tag{5}$$

where k and x stand for the 4-vectors $(x^0=t, x^1x^2x^3)$ and $(k^0, k^1k^2k^3)$; $kx=k^0x^0-k^1x^1-k^2x^2-k^3x^3=k^0x^0-\mathbf{kx}$ is the Minkowski scalar product and $d(k)=dk^0\, dk^1\, dk^2\, dk^3$ the Minkowski volume measure. Any solution ψ of the wave equation is concentrated on the light cone $k_0^2-|\mathbf{k}|^2=0$ in the Minkowski space. The wave frequency is $k_0=+|\mathbf{k}|=\omega$ in the upper cone and $k_0=-|\mathbf{k}|=-\omega$ in the lower cone.

The square-integrable solutions $\tilde{\psi}(k)$ satisfy the condition

$$\int_{k_0^2=\omega^2}\frac{d(\mathbf{k})}{\omega}\,|\tilde{\psi}(k)|^2<+\infty$$

and form a Hilbert Space $\mathscr{H}$ with respect to the relativistically invariant scalar product:

$$\langle\tilde{\psi}_1, \tilde{\psi}_2\rangle=\int\frac{d(\mathbf{k})}{\omega}\,\tilde{\psi}_1^*(\omega, \mathbf{k})\tilde{\psi}_2(\omega, \mathbf{k})+\int\frac{d(\mathbf{k})}{\omega}\,\tilde{\psi}_1^*(-\omega, \mathbf{k})\tilde{\psi}_2(-\omega, \mathbf{k}) \tag{6}$$

The Hilbert Space $\mathscr{H}$ is the direct sum of the Hilbert spaces $\mathscr{H}_+$ and $\mathscr{H}_-$, which carry the two mass-zero, helicity-zero irreducible representations of the Poincaré group. The ten anti-Hermitian generators of the Poincaré group are given as

$$P_0\tilde{\psi}(k)=-ik_0\tilde{\psi}(k) \tag{7}$$

$$P^a\tilde{\psi}(k)=-ik^a\tilde{\psi}(k) \tag{8}$$

$$J^a\tilde{\psi}(k)=-(k^b\,\partial_c-k^c\,\partial_b)\tilde{\psi}(k) \tag{9}$$

$$N^a\tilde{\psi}(k)=-k_0\,\partial_a\tilde{\psi}(k) \tag{10}$$

P^μ, $\mu=0, 1, 2, 3$, are the generators of space-time translation; J^a, $a=1, 2, 3$, are the generators of rotations; and N^a, $a=1, 2, 3$, are the generators of Lorentz boosts.

We remark that the representation space $\mathscr{H}$ provides a spectral representation for the unitary group U_t:

$$U_t\tilde{\psi}(k)=e^{-ik_0t}\tilde{\psi}(k) \tag{11}$$

The ten generators P^μ, J^a, N^a satisfy the commutation relations of the Poincaré Lie algebra

$$[P^\mu, P^\nu]=0 \tag{12}$$

$$[J^a, J^b]=\varepsilon_{abc}J^c \tag{13}$$

$$[N^a, N^b]=-\varepsilon_{abc}N^c \tag{14}$$

$$[J^a, N^b]=\varepsilon_{abc}N^c \tag{15}$$

$$[J^a, P_0]=0 \tag{16}$$

$$[N^a, P_0]=-P^a \tag{17}$$

$$[J^a, P^b]=\varepsilon_{abc}P^c \tag{18}$$

$$[N^a, P^b]=-\delta_{ab}P_0 \tag{19}$$

here

$$\varepsilon_{abc}=\begin{cases} 1, & abc \text{ even permutation of } 123 \\ -1, & abc \text{ odd permutation of } 123 \\ 0, & \text{otherwise} \end{cases}$$

is the Levi–Civita tensor, and

$$\delta_{ab}=\begin{cases} 0, & a\neq b \\ 1, & a=b \end{cases}$$

is the Kronecker delta.

The massless relativistic dynamical systems are characterized by the condition

$$(P_0)^2=(P^1)^2+(P^2)^2+(P^3)^2\equiv|\mathbf{P}|^2$$

A Hermitian time operator for the unitary evolution group $U_t=e^{P_0t}$ of any massless relativistic dynamical system may be constructed from the generators N^a, P^a of the Poincaré group:

$$T_R=-\tfrac{1}{2}\sum_{a=1}^{3}(N^aP^a|\mathbf{P}|^{-2}+|\mathbf{P}|^{-2}P^aN^a) \tag{20}$$

The expression (20) is found after observing that

$$\left[P_0, \sum_{a=1}^{3} N^aP^a\right]=\sum_{a=1}^{3}[P_0, N^a]P^a=(P^1)^2+(P^2)^2+(P^3)^2=|\mathbf{P}|^2$$

Therefore the operator $-\sum_{a=1}^{3} N^a P^a |\mathbf{P}|^{-2}$ satisfies the canonical commutation relation

$$\left[P_0, -\sum_{a=1}^{3} N^a P^a |\mathbf{P}|^{-2}\right] = -I$$

and (20) arises after symmetrization. The concrete form of the operator T_R on the space $\mathscr{H}$ of solutions of the wave equation is

$$T_R \tilde{\psi} = i \frac{\partial}{\partial k_0} \tilde{\psi} + \frac{i}{2} \frac{1}{k_0} \tilde{\psi} \tag{21}$$

This formula is found by replacing the forms (8) and (10) for the generators P^a and N^a in (20) and using the identity

$$\sum_{a=1}^{3} \frac{k^a}{k_0} \partial_a \tilde{\psi} = \frac{\partial}{\partial k_0} \tilde{\psi} \tag{22}$$

The identity (22) is easily proved from the fact that

$$k_0 = \pm [(k^1)^2 + (k^2)^2 + (k^3)^2]^{1/2}$$

A suitable domain $\mathscr{D}_R$ for the time operator T_R consists of all rapidly decreasing, infinitely differentiable functions $\psi(k)$ of the Schwartz class $\mathscr{S}$ on R^4, concentrated on the light cone, $k_0^2 = |\mathbf{k}|^2$, which satisfy the condition $\tilde{\psi}(k) = 0$ in some neighborhood of the point $\mathbf{k} = \mathbf{0}$. The Hermitian time operator T_R and the Poincaré generators P^μ, J^a, N^a are defined in $\mathscr{D}_R$ and do not lead out of $\mathscr{D}_R$. The domain $\mathscr{D}_R$ is also dense in $\mathscr{H}$ and invariant under the action of the Poincaré group. However, the time operator has to be self-adjoint. We have therefore to show that the densely defined Hermitian operator T_R has a self-adjoint extension T with domain $\mathscr{D}$ and also that the extension T satisfies the internal time property (1) in the domain $\mathscr{D}$. We shall show this by observing that the pair (U_t, T_R) is unitary equivalent to the canonical pair $(e^{-ik_0 t}, i\,\partial/\partial k_0)$ in another spectral representation of U_t, which we shall construct. This representation is carried in the Hilbert space $\mathscr{L}^2_{(-\infty+\infty)\times S}$ of square-integrable functions $f(k_0, \boldsymbol{\eta})$ on the unit sphere S. Here $\boldsymbol{\eta}$ is the unit vector in the direction of the propagation characterized by the spherical coordinates (θ, ϕ). The scalar product on $\mathscr{L}^2_{(-\infty+\infty)\times S}$ is

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} dk_0 \int d(\boldsymbol{\eta}) f^*(k_0, \boldsymbol{\eta}) g(\mathbf{k}_0, \boldsymbol{\eta}) \tag{23}$$

with $d(\boldsymbol{\eta}) = d\theta \sin\theta\, d\phi$.

The unitary transformation V of $\mathscr{H}$ onto $\mathscr{L}^2_{(-\infty+\infty)\times S}$ is

$$V: \quad \tilde{\psi} \mapsto f = V\tilde{\psi}$$

$$f(k_0, \boldsymbol{\eta}) = V\tilde{\psi}(k_0, \boldsymbol{\eta}) = \omega^{1/2}\tilde{\psi}(k)$$

The unitarity of V follows immediately:

$$\begin{aligned}
\langle \tilde{\psi}_1, \tilde{\psi}_2 \rangle &= \int \frac{d(\mathbf{k})}{\omega} \tilde{\psi}_1^*(\omega, \mathbf{k})\tilde{\psi}_2(\omega, \mathbf{k}) + \int \frac{d(\mathbf{k})}{\omega} \tilde{\psi}_1^*(-\omega, \mathbf{k})\tilde{\psi}_2(-\omega, \mathbf{k}) \\
&= \int_0^{+\infty} d\omega \int d(\boldsymbol{\eta})\, \omega\tilde{\psi}_1^*(\omega, \boldsymbol{\eta}\omega)\tilde{\psi}_2(\omega, \boldsymbol{\eta}\omega) \\
&\quad + \int_0^{\infty} d\omega \int d(\boldsymbol{\eta})\, \omega\tilde{\psi}_1^*(-\omega, \boldsymbol{\eta}\omega)\tilde{\psi}_2(-\omega, \boldsymbol{\eta}\omega) \\
&= \int_{-\infty}^{+\infty} dk_0 \int d(\boldsymbol{\eta})(V\tilde{\psi}_1)^*(k_0, \boldsymbol{\eta})(V\tilde{\psi}_2)(k_0, \boldsymbol{\eta})
\end{aligned}$$

The unitary group U_t on $\mathscr{H}$ has the same form in $\mathscr{L}^2_{(-\infty+\infty)\times S}$:

$$VU_tV^{-1}f(k_0, \boldsymbol{\eta}) = e^{-ik_0t}f(k_0, \boldsymbol{\eta})$$

while the time operator T_R is transformed into the canonical form $i\,\partial/\partial k_0$:

$$\begin{aligned}
VT_RV^{-1}f(k_0, \boldsymbol{\eta}) &= \omega^{1/2}\left(i\frac{\partial}{\partial k_0} + \frac{i}{2}\frac{1}{k_0}\right)\omega^{-1/2}f(k_0, \boldsymbol{\eta}) \\
&= \omega^{1/2} i \frac{\partial}{\partial k_0}[\omega^{-1/2}f(k_0, \boldsymbol{\eta})] + \frac{i}{2}\frac{1}{k_0}f(k_0, \boldsymbol{\eta}) \\
&= i\frac{\partial}{\partial k_0}f(k_0, \boldsymbol{\eta})
\end{aligned}$$

The domain $\mathscr{D}_R$ is mapped onto the domain $V[\mathscr{D}_R]$, which consists of all Schwartz functions $f(k_0, \boldsymbol{\eta})$ on $(-\infty, +\infty) \times S$ which vanish in some neighborhood of $k_0 = 0$.

The domain $V[\mathscr{D}_R]$ is dense and invariant under the unitary transformations $U_t = e^{-ik_0t}$. The operator $i\,\partial/\partial k_0$ on $V[\mathscr{D}_R]$ has the self-adjoint extension $T_\Sigma = i\,\partial/\partial k_0$ with domain $\mathscr{D}_\Sigma$, which consists of all absolutely continuous square-integrable functions f with square-integrable derivatives f'.

The domain $\mathscr{D}_\Sigma$ is invariant under the unitary group $U_t = e^{-ik_0t}$ and it can be directly verified that the operator $T_\Sigma = i\,\partial/\partial k_0$ satisfies the internal time property (1) on $\mathscr{D}_\Sigma$. Because of the unitary equivalence V, we conclude that the time operator T_R on $\mathscr{D}_R \subseteq \mathscr{H}$ has the self-adjoint extension $T = V^{-1} T_\Sigma V$ with domain $\mathscr{D} = V^{-1}[\mathscr{D}_\Sigma]$ and that T satisfies the internal time property (1) with $U_t = e^{-ik_0 t}$.

The internal time property (1) is equivalently expressed in terms of the spectral projections $\mathbb{P}_\tau$, $\tau \in \mathbb{R}$, of the operator T as the imprimitivity condition:

$$\mathbb{P}_{t+\tau} = U_t \mathbb{P}_\tau U_{-t} \tag{24}$$

$$T = \int_{-\infty}^{+\infty} \tau \, d\mathbb{P}_\tau \tag{25}$$

The imprimitivity condition arises simply by inserting the expression (25) into the internal time property (1). Because of the imprimitivity condition the range $\mathscr{H}_0$ of the spectral projection $\mathbb{P}_0$ of the time operator T is an incoming subspace as defined by Lax and Phillips (1989) in the context of the scattering theory of wave equations. In particular the projection $\mathbb{P}_0$ of the time operator $T_\Sigma = i\,\partial/\partial k_0$ in the spectral representation is the projection onto the upper Hardy–Lebesque space of functions $f(k_0, \boldsymbol{\eta})$ which are the Fourier transforms of functions with support $(-\infty, 0)$. This can be seen from the fact that the differentiation operator is unitary equivalent to the multiplication operator through the Fourier–Plancherel transformation (see, for example, Akhiezer and Glazman, 1981). Therefore the upper Hardy–Lebesque space is an incoming subspace with respect to the evolution group $U_t = e^{-ik_0 t}$ in the spectral representation of the wave equation. This incoming subspace is the same with the Lax and Phillips (1989) incoming subspace which they constructed on the basis of Huygens' principle. [For the Lax–Phillips scattering theory see also Reed and Simon (1979), Vol. 3.]

We are therefore able to construct incoming subspaces for free wave equations from the algebraic construction of the time operator, without invoking Huygens' principle. In fact, incoming subspaces can be constructed from the time operator in the cases of one and two space dimensions where Huygens' principle does not hold. The internal time operator for the one- and two-dimensional wave equation is given in Appendix A. However, this method is not applicable to wave equations with rigid obstacles, as the notion of rigid obstacle is not relativistic.

3. THE RELATIVISTIC INTERNAL TIME ALGEBRA

The algebra of relativistic systems with internal time is the Lie algebra generated by the ten generators P^μ, J^a, N^a of the Poincaré group satisfying the commutation relations (12)–(19) and the internal time T given by (20).

The commutation relations of the internal time with the ten generators of the Poincaré group are

$$[P_0, T] = -I \tag{26}$$

$$[P^a, T] = -P^a P_0^{-1} \equiv -V^a \tag{27}$$

$$[J^a, T] = 0 \tag{28}$$

$$[N^a, T] = TP^a P_0^{-1} \equiv Q^a \tag{29}$$

These commutation relations can be verified directly for the wave equation by replacing the explicit forms (7)–(10) for the Poincaré generators and the form (21) for the time operator using the identity (22). A general proof for massless relativistic systems of any helicity is given in Appendix B. The commutation relations may also be proved by noting that the time operator (20) may be expressed in terms of the dilatations generator D. The relevant formulas are given in Appendix D.

The commutation relations (27) and (29) show that the internal time does not commute with the generators of translations and Lorentz boosts N^a and that two new entities $V^a = P^a P_0^{-1}$ and $Q^a = TV^a$ appear which generate the infinite number of entities $V^a V^b$, $V^a V^b V^c, \ldots$ and $TV^a V^b$, $TV^a V^b V^c, \ldots$ as shown by the commutation relations of V^a and Q^a:

$$[P_0, V^a] = 0 \tag{30}$$

$$[P^a, V^b] = 0 \tag{31}$$

$$[J^a, V^b] = \varepsilon_{abc} V^c \tag{32}$$

$$[N^a, V^b] = V^a V^b - \delta_{ab} I \tag{33}$$

$$[T, V^a] = 0 \tag{34}$$

$$[V^a, V^b] = 0 \tag{35}$$

$$[P_0, Q^a] = -V^a \tag{36}$$

$$[P^a, Q^b] = -V^a V^b \tag{37}$$

$$[J^a, Q^b] = \varepsilon_{abc} Q^c \tag{38}$$

$$[N^a, Q^b] = 2TV^a V^b - \delta_{ab} T \tag{39}$$

$$[T, Q^b] = 0 \tag{40}$$

$$[V^a, Q^b] = 0 \tag{41}$$

$$[Q^a, Q^b] = 0 \tag{42}$$

The entities V^aV^b and TV^aV^b, in turn, generate $V^aV^bV^c$ and $TV^aV^bV^c$, as shown in the following commutation relations, which are the only nonzero commutation relations of V^aV^b, TV^aV^b:

$$[J^a, V^bV^c] = \varepsilon_{abc'}V^cV^{c'} + \varepsilon_{acb'}V^bV^{b'} \tag{43}$$

$$[N^a, V^bV^c] = 2V^aV^bV^c - \delta_{ab}V^b - \delta_{ac}V^c \tag{44}$$

$$[P_0, TV^bV^c] = -V^bV^c \tag{45}$$

$$[P^a, TV^bV^c] = -V^aV^bV^c \tag{46}$$

$$[J^a, TV^bV^c] = \varepsilon_{abc'}TV^cV^{c'} + \varepsilon_{acb'}V^bV^{b'} \tag{47}$$

$$[N^a, TV^bV^c] = 3TV^aV^bV^c - \delta_{ab}Q^b - \delta_{ac}Q^c \tag{48}$$

The commutation relations (30)–(48) follow easily from (26)–(29) and the Poincaré algebra.

The commutators of the monomials $V^aV^b \ldots$ and $TV^aV^b \ldots$ and the boost generators N^a always give rise to monomials of degree greater by one and thus the infinite commutative algebras $\Pi_{[V]}$ and $T\Pi_{[V]}$ of polynomials in velocities V^a emerge.

The relativistic internal time algebra includes the Poincaré algebra with commutation relations (12)–(19) and the monomials $V^aV^b \ldots$, and $TV^aV^b \ldots$ with commutation relations (26)–(48). The form of the commutation relations shows clearly that the relativistic internal time algebra is the semidirect sum Lie algebra

$$\mathscr{P}_4 \oplus\!\!\!\!\!\!\supset (\Pi_{[V]} \oplus T\Pi_{[V]}) \tag{49}$$

or

$$\mathscr{L}_4 \oplus\!\!\!\!\!\!\supset [\mathscr{T}_4 \oplus\!\!\!\!\!\!\supset (\Pi_{[V]} \oplus T\Pi_{[V]})] \tag{50}$$

where $\mathscr{P}_4$ and $\mathscr{L}_4$ stand for the Poincaré and Lorentz lie algebras, respectively.

Expression (49) shows that the RIT algebra is an extension of the Poincaré algebra by the infinite commutative algebra $\Pi_{[V]} \oplus T\Pi_{[V]}$. The infinite Lie algebra $\mathscr{X} = \mathscr{T}_4 \oplus\!\!\!\!\!\!\supset (\Pi_{[V]} \oplus T\Pi_{[V]})$ in expression (50) is nilpotent of order two with center $\Pi_{[V]}$. For the relevant Lie-algebraic definitions see, for example, Barut and Raczka (1977).

The semidirect sum structure of the relativistic internal time algebra means that the infinite algebra $\mathscr{X}$ is a representation space for the Lorentz algebra $\mathscr{L}_4$. Therefore the characterization of this representation by means of the Gelfand–Naimark numbers provides a characterization of the relativistic internal time algebra and gives the possibility for comparison with infinite Lie algebras with the same structure, such as the Bondi–Metzner–Sachs algebra. The two Lie algebras turn out to be different and the method also reveals some minor errors in the commutation relations of the BMS algebra (Antoniou and Misra, 1991).

The relativistic internal time algebra commutation relations (12)–(19) and (26)–(48) are abstract and formal. In any representation of this algebra the ten generators of the Poincaré group are taken to be anti-Hermitian operators, while the internal time T and the monomials $V^a V^b \ldots$, $TV^a V^b \ldots$ are Hermitian operators. Furthermore, the ten Poincaré generators and the time operator have to be defined in a common dense domain $\mathscr{D}_R$ and should not lead out of this domain. This is indeed true for the helicity-zero system (the wave equation), as has been shown in Section 2. For higher-helicity systems similar arguments apply for the domain of the algebra.

For higher-helicity systems the time operator and the relativistic internal time algebra satisfy the supplementary conditions associated with the dynamical equations. The helicity-one system, for example, is the electromagnetic radiation field in free space described by the Maxwell equations.

The Maxwell equations can be expressed in terms of the vector potential A^μ, $\mu = 0, 1, 2, 3$, as the wave equations

$$\partial_t^2 A^\mu = \Delta A^\mu$$

together with the Lorentz gauge condition:

$$\partial_\mu A^\mu = 0$$

It can be verified directly (Antoniou, 1988) that the time operator is compatible with the Lorentz condition, namely

$$\partial_\mu (TA^\mu) = 0 \qquad \text{if} \quad \partial_\mu A^\mu = 0$$

Similarly, the helicity-two system is the linear gravitational field in flat space-time. The linear deviation from the Minkowski metric is the symmetric 2-tensor field $h^{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$, satisfying the wave equations

$$\partial_t^2 h^{\mu\nu} = \Delta h^{\mu\nu}$$

together with the De Donder harmonic condition

$$\partial_\nu h^{\mu\nu} = 0$$

In this case also, the time operator is compatible (Antoniou, 1988) with the De Donder condition.

It is well known that the physically observed integer-helicity systems, namely the scalar waves, the free electromagnetic radiation field and the linear gravitational field in flat space-time, are real fields. Therefore the relativistic internal time algebra has to preserve the reality of these fields. This is indeed the case, as can be directly verified from the explicit forms of the Poincaré generators and the time operator. In the case of the wave equation in the Fourier representation, the reality condition is expressed as

$$\tilde{\psi}^*(-k)=\tilde{\psi}(k) \tag{51}$$

4. ON THE MEANING OF THE COMMUTATION RELATIONS

The commutator $[J^a, T]=0$ in (28) shows that the age of the system does not change under rotations. However, the age of the system changes under space translations, as the commutator (27) shows, a fact expected for any massless relativistic system: $(P_0)^2=(P^1)^2+(P^2)^2+(P^3)^2$. For if $[P^a, T]=0$, then we would have $[P_0, T]=0$, contrary to the canonical commutation relation $[P_0, T]=-I$.

The internal time T generates via (27) the velocity operator $V^a=P_0^{-1}P^a$ of the system. Relation (32) shows that the velocity V^a transforms as a vector under space translations and relation (33) is an expression of the Einstein velocity transformation formula under Lorentz boosts with velocities $v^a=c\text{th}\,\zeta^a$,

$$\begin{aligned}
[N^a, V^a] &= \frac{d}{d\zeta^a}\left(e^{\zeta^a N^a} V^a\, e^{-\zeta^a N^a}\right)_{\zeta^a=0} \\
&= \frac{d}{d\zeta^a}\left(\frac{V^a-\text{th}\,\zeta^a\cdot c}{1-(\text{th}\,\zeta^a V^a)/c}\right)_{\zeta^a=0} \\
&= \frac{1}{c}\,V^a V^a - cI \\
[N^b, V^a] &= \frac{d}{d\zeta^a}\left(\frac{V^a-(\text{ch}\,\zeta^a)^{-1}}{1-(\text{th}\,\zeta^a V^b)/c}\right)_{\zeta^a=0} \\
&= \frac{1}{c}\,V^a V^b
\end{aligned}$$

The commutation relation (33) is the natural property of the relativistic velocity operator (Jordan, 1977) and brings into the algebra the nonlinearity which generates the infinite elements. This feature of the velocity operators was also mentioned by Durant (1973).

The transformation of the internal time under Lorentz boosts is expressed in the commutation relation (29). For example, under a Lorentz boost with velocity $v^1 = \text{cth}\, \zeta^1$,

$$e^{\zeta^1 N^1} T e^{-\zeta^1 N^1} = \frac{1}{\gamma(v^1)} \frac{T}{1 - v^1 V^1/c^2} \tag{52}$$

with

$$\gamma(v^1) = \frac{1}{[1-(v^1/c)^2]^{1/2}}$$

For the proof of formula (52) and of the following formulas see Appendix C.

As mentioned in the Introduction, the internal time allows us to attribute an average age to field states. According to formula (52), the average age of the boosted state is different from what one would expect from Einstein's time transformation formulas. This difference reflects the fact that the internal time operator has nonlocal character, as it involves the operator $|\mathbf{P}|^{-2}$, which is nonlocal when expressed in spatial coordinates.

Formula (29) also shows that the internal time generates with the boost generators the new entity $Q^a = TV^a$ to be identified with a kind of internal position observable of the relativistic system. Relation (36) shows simply that $\dot{Q}^a = V^a$ and relation (30) expresses that we have a free system: $\dot{V}^a = \ddot{Q}^a = 0$. The compatibility of the internal position observables Q^1, Q^2, Q^3 is expressed by their commutativity (42). Relation (38) shows that the observables Q^1, Q^2, Q^3 transform like the components of a 3-vector under space rotations. The transformation of the internal position operator Q^a under Lorentz boosts is expressed by (39). For example,

$$e^{\zeta^1 N^1} Q^1 e^{-\zeta^1 N^1} = \frac{1}{\gamma(v^1)} \frac{Q^1 - Tv^1}{(I - v^1 V^1/c^2)^2} \tag{53}$$

This formula also differs from Einstein's space transformation formulas. The commutation relation (37) is not the expected Heisenberg relation of localizability,

$$[P^a, Q^b] = -\delta_{ab} I$$

Expressions involving $V^a V^b$ in the commutator of the position observable with the momentum have been discussed in various proposals for relativistic position observables (e.g., Pryce, 1948; Finkelstein, 1949; Johnson, 1969; Durant, 1973).

The internal position Q^a is related to the center of energy X^a through the formula

$$Q^a = |P_0|(\mathbf{X}\cdot\mathbf{V})X^a|P_0|^{-1} \tag{54}$$

The center of energy of a relativistic system is the symmetrized operator:

$$X^a = -\tfrac{1}{2}(N^a P_0^{-1} + P_0^{-1} N^a) \tag{55}$$

The Lorentz boost generator N^a is interpreted as the moment of the energy (energy times the position coordinate of the center of energy of the system), which replaces the center of mass of Galilei systems (Sudarshan and Mukunda, 1983; Pryce, 1948).

The center of energy is the Newton-Wigner (1949) position operator for massless elementary systems of helicity zero,

$$X^a = i\frac{\partial}{\partial k^a} - \frac{i}{2}\frac{k^a}{|\mathbf{k}|} \tag{56}$$

5. CONCLUDING REMARKS

The time and position operators T, Q^a, $a = 1, 2, 3$, of relativistic fields provide an internal space-time arising from the intrinsic dynamics of the field as a whole. The transformation laws (52) and (53) of T and Q^a under Lorentz boosts are radically different from the corresponding formulas of Einstein. Therefore the space-time corresponding to the internal time and position is not the Minkowski space-time of events labels of the observer's registrations. The tetrad T, Q^a is not a 4-vector. Attempts to construct 4-vector relativistic position operators suffer from other difficulties, such as lack of Hermiticity and/or mutual noncompatibility; see, for instance, the review of Kalnay (1971).

The algebra of relativistic systems with internal time is an infinite-dimensional extension of the Poincaré algebra by the infinite commutative algebra $\Pi_{[V]} \oplus T\Pi_{[V]}$ of the monomials $V^a V^b \ldots$ and $TV^a V^b \ldots$. Since the massless relativistic fields carry representations of the algebra, one expects that the massless fields have a larger symmetry group than just the 15-parameter conformal group. These additional symmetries correspond to the generators $V^a V^b \ldots$, $TV^a V^b \ldots$; they are nonlocal and they cannot be implemented by point transformations of Minkowski space-time, in contrast to the Poincaré and conformal symmetries. This new type of nonlocal internal symmetry of fields associated with the internal time does not seem to be related to any known type of internal symmetry. The nonlocal symmetries refer to the field itself; they are intrinsic properties of the field and not of the operational Minkowski continuum of events-labels of the infinite degrees of freedom of the field.

It is clear that formula (20) defines a formal time operator T_m for the massive Klein–Gordon field as well. The operator T_m acts on the Hilbert space of square-integrable solutions of the Klein–Gordon equation. The operator T_m is densely defined on the domain $\mathscr{D}_m$, which consists of all functions $\tilde{\psi}(k)$ which are concentrated on the two mass hyperboloids $(k_0)^2 - |\mathbf{k}|^2 = m^2$ and vanish in some neighborhood of each of the two points $(0, m)$ and $(0, -m)$.

The operator T_m has self-adjoint extensions because it commutes with the conjugation operator according to von Neumann's theorem (see, for example, Reed and Simon, 1979, Vol. 2, Theorem X.3),

$$C\tilde{\psi}(k_0, \mathbf{k}) = \tilde{\psi}^*(-k_0, \mathbf{k}) \tag{57}$$

However, none of the self-adjoint extensions of T_m can satisfy the internal time property (1), because the spectrum of the generator of the unitary group U_t is not the entire real line and it is well known that if U_t admits a self-adjoint time operator, the spectrum of the (self-adjoint) generator of U_t must be the entire real line.

Let us also mention that the commutation relations of the corresponding algebra for the massive systems have a more complicated structure.

APPENDIX A. TIME OPERATORS FOR THE WAVE EQUATION IN ONE AND TWO DIMENSIONS

The internal time (20) for the one-dimensional wave equation is

$$T_R = -\tfrac{1}{2}[N^1 P^1 (P^1)^{-2} + (P^1)^{-2} P^1 N^1] \tag{A1}$$

After substituting the expressions (8), (10) for the generators N^1, P^1 we obtain

$$T_R \tilde{\psi} = i \frac{\partial \tilde{\psi}}{\partial k_0} - \frac{i}{2k_0} \tilde{\psi} \tag{A2}$$

The operator (A2) is unitary equivalent to the canonical form under the unitary transformation V:

$$V\tilde{\psi} = |k^1|^{-1/2} \tilde{\psi} \tag{A3}$$

The unitarity of V follows immediately as in the three-dimensional case:

$$\langle \tilde{\psi}_1, \tilde{\psi}_2 \rangle = \int \frac{d(k^1)}{|k^1|} \tilde{\psi}_1^* \tilde{\psi}_2 = \int d(k^1) \, (|k^1|^{-1/2} \tilde{\psi}_1)^* (|k^1|^{-1/2} \tilde{\psi}_2)$$

For the two-dimensional wave equation the internal time (20) is in the canonical form:

$$T_R \tilde{\psi} = i \frac{\partial}{\partial k_0} \tilde{\psi} \tag{A4}$$

Therefore there is no need for any unitary transformation. The Hilbert space $\mathscr{H}$ provides the canonical spectral representation for the two-dimensional wave equation.

APPENDIX B

The commutation relations (26)–(28) follow immediately after observing that the internal time (20) may be written as

$$T = -(N^a P^a + \tfrac{1}{2} P_0) |\mathbf{P}|^{-2} \tag{B1}$$

Formula (B1) follows with the help of the formulas

$$P^a N^a = N^a P^a + 3 P_0 \tag{B2}$$

$$[N^a, |\mathbf{P}|^{-2}] = 2 P_0 P^a |\mathbf{P}|^{-4} \tag{B3}$$

The proof of formula (29) is more involved:

$$\begin{aligned}
[N^1, T] &= [N^1, -(N^a P^a + \tfrac{1}{2} P_0) |\mathbf{P}|^{-2}] \\
&= -[N^1, N^a P^a] |\mathbf{P}|^{-2} - \tfrac{1}{2}[N^1, P_0] |\mathbf{P}|^{-2} \\
&\quad - N^a P^a [N^1, |\mathbf{P}|^{-2}] - \tfrac{1}{2} P_0 [N^1, |\mathbf{P}|^{-2}] \\
&= -(J^2 P^3 - J^3 P^2 - N^1 P_0) |\mathbf{P}|^{-2} + \tfrac{1}{2} P^1 |\mathbf{P}|^{-2} \\
&\quad - 2 N^a P^a P^1 P_0 |\mathbf{P}|^{-4} - (P_0)^2 P^1 |\mathbf{P}|^{-4}
\end{aligned}$$

We express J^2 and J^3 in terms of the Pauli–Lubanski–Bargmann vectors W^2, W^3:

$$\begin{aligned}
J^2 &= (W^2 - N^3 P^1 + N^1 P^3) P_0^{-1} \\
J^3 &= (W^3 - N^1 P^2 + N^2 P^3) P_0^{-1}
\end{aligned} \tag{B4}$$

After the calculations we get

$$\begin{aligned}[N^1, T] = &-[N^a P^a P^1 P_0^{-1} + (W^2 P^3 - W^3 P^2) P_0^{-1} \\ &+ N^1 P_0 (|\mathbf{P}|^2 P_0^{-2} - I)] |\mathbf{P}|^{-2} \\ &- 2 N^a P^a P^1 P_0 |\mathbf{P}|^{-4} - \tfrac{1}{2} P_0 P^1 |\mathbf{P}|^{-2} (2 P_0^2 |\mathbf{P}|^{-2} - I)\end{aligned}$$

It is well known (Bargmann and Wigner, 1948) that for all massless systems the Pauli–Lubanski–Bargmann vector is proportional to the momentum vector; therefore

$$W^2 P^3 - W^3 P^2 = 0$$

Using the fact that $|\mathbf{P}|^2 = (P_0)^2$, we get (29)

$$\begin{aligned}[N^1, T] &= -N^a P^a P^1 P_0^{-1} |\mathbf{P}|^{-2} - 2 N^a P^a P^1 P_0 |\mathbf{P}|^{-4} - \tfrac{1}{2} P_0 P_1 |\mathbf{P}|^{-2} \\ &= -(N^a P^a + \tfrac{1}{2} P_0) |\mathbf{P}|^{-2} P^1 P_0^{-1} \\ &= T P^1 P_0^{-1}\end{aligned}$$

APPENDIX C

Formula (52) can be proved as follows. From the Baker–Campell–Hausdorff formula

$$e^{\zeta^1 N^1} T e^{-\zeta^1 N^1} = T + \zeta^1 [N^1, T] + \frac{(\zeta^1)^2}{2!} [N^1, [N^1, T]] + \cdots \tag{C1}$$

and the commutation relations of T, we conclude that the right-hand side of (C1) should be of the form $Tf(V^1)$. The function $f(V^1)$ can be found from the fact that the commutation relation $[P_0, T] = -I$ is preserved under Lorentz boosts:

$$\begin{aligned}&[e^{\zeta^1 N^1} P_0 e^{-\zeta^1 N^1}, e^{\zeta^1 N^1} T e^{-\zeta^1 N^1}] = -I \\ &\Leftrightarrow [P_0 \operatorname{ch} \zeta^1 - P^1 \operatorname{sh} \zeta^1, Tf(V^1)] = -I \\ &\Leftrightarrow -f(V^1) \operatorname{ch} \zeta^1 + \frac{1}{c} V^1 f(V^1) \operatorname{sh} \zeta^1 = -I \\ &\Leftrightarrow f(V^1) = \frac{1}{\operatorname{ch} \zeta^1 - \operatorname{sh} \zeta^1 (1/c) V^1} = \frac{1}{\gamma(v^1)(I - v^1 V^1 / c^2)}\end{aligned}$$

Formula (53) follows from (52) and the velocity operator transformation formula. Formula (54) follows easily using the lemma

$$|P_0|X^a|P_0|^{-1}=X^a-V^aP_0^{-1} \tag{C2}$$

Formula (C2) follows from the commutator

$$[P_0, X^a]=-V^a \tag{C3}$$

APPENDIX D. TIME OPERATOR AND THE DILATATION GENERATOR

The time operator (20) may be written as

$$T=-\tfrac{1}{2}(N^aP^a|\mathbf{P}|^{-2}P_0+P_0|\mathbf{P}|^{-2}P^aN^a)+\tfrac{1}{2}P_0^{-1} \tag{D1}$$

or

$$T=DP_0^{-1}+\tfrac{1}{2}P_0^{-1} \tag{D2}$$

with

$$D=-\tfrac{1}{2}(N^aP^a|\mathbf{P}|^{-2}P_0+P_0|\mathbf{P}|^{-2}P^aN^a) \tag{D3}$$

The operator (D3) satisfies the commutation relations of the dilatation generator for massless systems, namely

$$[D, P^\mu]=P^\mu \tag{D4}$$

$$[D, J^a]=0 \tag{D5}$$

$$[D, N^a]=0 \tag{D6}$$

Formulas (D1), (D4), and (D5) follow immediately. The proof of formula (D6) is more involved and it is based on the Pauli–Lubanski–Bargmann vector following an argument similar to the discussion of formula (29) in Appendix B. Formula (D6) may also be proved from (29) and (D2). The properties (D4)–(D6) of the operator (D3) were also proved by Fushchich and Nikitin (1978), using explicit forms of the Poincaré generators in the space of solutions of massless relativistic equations. However, our proof does not depend on any concrete representation of the Poincaré algebra.

The commutation relations (26)–(29) follow immediately from (D2) and (D4)–(D6) together with the commutation relations of the Poincaré algebra (12)–(16).

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